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MIXED PROBLEM OF CRACK THEORY FOR ANTIPLANE DEFORMATION

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The elastic equilibrium of an isotropic plane with one linear defect under conditions of longitudinal shear is considered. The strain field is constructed by the solution of a two-dimensional boundary-value Riemann problem with variable coefficients. A special method that reduce the general two-dimensional problem to two one-dimensional problems is proposed. The strain field is described by three types of asymptotic relations: for the tips of the defect, for the tips of the reinforcing edge, and also at a distance from the closely spaced tips of the defect and the rib. The general form of asymptotic relations for strains with finite energy is deduced from analysis of the variational symmetries of the equations of longitudinal shear. A paradox of the primal mixed boundary-value problem for cracks is formulated and a method of solving the problem is proposed.

Mixed problems of crack theory have been insufficiently investigated at present. They have not been adequately addressed in known reference books [1, 2], although they are of both applied and theoretical interest. A peculiarity of mixed problems for cracks, as will be shown, is the occurrence of unstable solutions that have paradoxical asymptotic expressions for internal force factors. At the tip of a defect, they can have any singularity from 0 to -1. The paradoxicality of such asymptotic expressions is due to their incompatibility with the variational symmetry of antiplane strain equations about the group of spatial shears that uniquely determines the classical form of asymptotic distribution of strain. It is shown that, under conditions of limited strength, it is impossible to attain defect shapes that lead to the indicated singular solutions. As a result, the paradoxical asymptotic expressions (which include solutions of the primal mixed boundary-value problem with singularities of orders of $-1/4$ and $-3/4$) can be used only to describe strains far away from the closely spaced tips of the defect and the rib, at which other asymptotic expressions are valid. At the same time, they can be used to estimate the strain-intensity coefficients at the tip of the defect.

1. Boundary-Value Problem. We consider the elastic equilibrium of an isotropic plane under longitudinal-shear conditions. The plane is weakened by a rectilinear cut along the segment of the real axis $|x| < 1$. The segment $a < x < b$ on the lower side of the cut is reinforced by an infinitely thin elastic rib. By definition, the rib offers elastic resistance to shear strain only and does not resist tension–compression and bending. Such schematization is allowable if the rib consists of momentless elastic fibers (by the terminology of [3, 4]) that do not interact with one another and are directed at an angle of 45° to the plane xOy.

At infinity, the plane is loaded by a uniform load $\gamma^0_{-1} \equiv \gamma^0_{13} - i\gamma^0_{23}$. The sides of the defect are free from external forces, and rotation of the rib in its plane is prevented by internal moments (Fig. 1).

Static equilibrium of the plane elements located on the sides of the cut is ensured by conditions of the form

$$\gamma^+_{23}(x,0) = 0, \quad (\chi_1 + \chi_3)\gamma^-_{23} + \chi_2\gamma^-_{\lambda 3} = 0, \quad \chi_k(x) = \begin{cases} 1, & x \in L_k, \\ 0, & x \notin L_k, \end{cases}$$
$$\gamma_{\lambda 3} \equiv \alpha\gamma_{23} + \beta\gamma_{13}, \quad \alpha + i\beta \equiv e^{+i\lambda}, \quad \alpha = (1 + (c/\mu)^2)^{-1/2}, \quad \beta = \alpha c/\mu, \tag{1.1}$$

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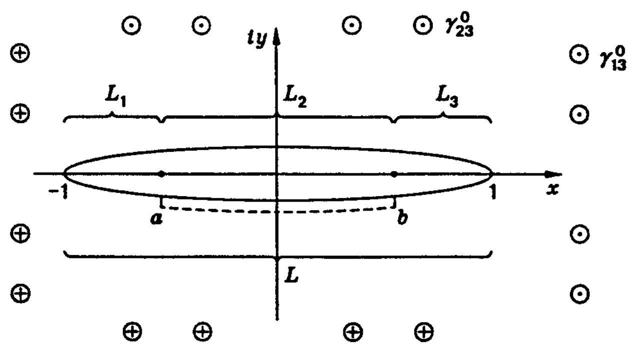

Fig. 1

where $(\gamma_{13}, \gamma_{23})$ are components of the shear-strain field, c is the shearing rigidity of the rib, μ is the shear modulus of the main material, and L_k $(k = 1, 2, 3)$ are the regions of the piecewise uniformity of the boundary conditions. The boundary conditions (1.1) combine the condition of a normal derivative on the upper side of the cut L and the lower sides of the lines L_1 and L_3 and the condition of an oblique derivative on the lower side of the contour L_2 (Fig. 1).

The strain field is conveniently represented as a superposition of the following three components:

$$\gamma_{13} - i\gamma_{23} = \gamma_{-1}^0 + \gamma_1(z) + i\gamma_2(z). \tag{1.2}$$

Here the unknown analytical functions $\gamma_1(z)$ and $\gamma_2(z)$ are in turn represented as Cauchy-type integrals with purely real densities. Calculating the limiting strains (1.2) on the contour of the defect and substituting them into boundary conditions (1.1), we come to the following boundary-value Riemann problem for the functions $\gamma_1(z)$ and $\gamma_2(z)$:

$$\begin{gathered}\gamma_1^+ + (\chi_1 + \chi_3)\gamma_1^- + \chi_2 e^{-i\lambda}(\cos\lambda\gamma_1^- + \sin\lambda\gamma_2^-) = 2i(\chi_1 + \chi_3)\gamma_{23}^0 + i\chi_2(\gamma_{23}^0 + e^{-i\lambda}\gamma_{\lambda 3}^0), \\ \gamma_2^+ - (\chi_1 + \chi_3)\gamma_2^- + \chi_2 e^{-i\lambda}(\cos\lambda\gamma_1^- - \sin\lambda\gamma_2^-) = \chi_2(\gamma_{23}^0 - e^{-i\lambda}\gamma_{\lambda 3}^0).\end{gathered} \tag{1.3}$$

Here $\gamma_{\lambda 3}^0 = \alpha\gamma_{23}^0 + \beta\gamma_{13}^0$ are the external strain in the oblique areas. In particular, the boundary-value problem for an ideal crack is obtained from relation (1.3) by three methods: $\chi_1 = 1$ and $\chi_2 = \chi_3 = 0$, or $\chi_3 = 1$ and $\chi_1 = \chi_2 = 0$, or $\lambda = 0$. The primal mixed boundary-value problem (a rigid inclusion with a detached upper side) corresponds to the parameters $\chi_1 = \chi_2 = 0$ and $\lambda = \pi/2$.

The additional condition of single-valued displacements and the absence of the principal vector force at the rib is written as the equality

$$\operatorname{Res}(\gamma_1 + i\gamma_2)\Big|_{z=\infty} = 0. \tag{1.4}$$

2. Solution of Problem (1.3). A distinctive feature of the two-dimensional boundary-value problem (1.3) is that its coefficient matrix is symmetric and is piecewise-constant. We construct its solution using a special method, which is much simpler than the general theoretical procedures described, e.g., in [5]. We briefly describe this method.

In matrix designations, problem (1.3) has the form

$$\gamma^+ - G\gamma^- = f. \tag{2.1}$$

We introduce a new vector of the unknowns, $\Gamma(z) = C(z)\gamma(z)$, where $C(z)$ is an orthogonal analytical matrix:

$$C(z) = \begin{bmatrix} \cos\omega(z) & -\sin\omega(z) \\ \sin\omega(z) & \cos\omega(z) \end{bmatrix}. \tag{2.2}$$

In the new unknowns, the boundary-value problem (2.1) is written as $\Gamma^+ - C^+G(C^-)^{-1}\Gamma^- = F$ and $F = C^+f$. The matrix $C(z)$ is chosen so that the new coefficient matrix becomes a diagonal matrix. In this case, the two-dimensional boundary-value problem is divided into two independent one-dimensional Riemann problems.

Carrying out the program described above, we come to the conclusion that on L the rotation angle $\omega(z)$ in matrix (2.2) should satisfy the boundary-value condition

$$\omega^+ + \omega^- = \omega_0(x), \quad x \in L, \quad \omega_0(x) = \pi(k\chi_1 + m\chi_3) + \chi_2(l\pi - \lambda), \quad k, l, m = 0, \pm 1, \pm 2, \ldots \tag{2.3}$$

The unique solution of the auxiliary boundary-value problem (2.3) is obtained under the following restrictions:

(1) At infinity, the function $\omega(z)$ can have an apparent singularity. In particular, for the primal mixed boundary-value problem, it follows from conditions (1.3) that $\omega(z) = -\pi/4$.

(2) In the vicinity of the tips of the defect, the strains depend monotonically on the polar radius and do not vary. This is possible if the function $\omega(z)$ is limited at these points.

(3) In the vicinity of the tips of the rigid reinforcing rib ($\lambda = \pi/2$), the order of the singularity of strains is the same as in the problem of a rigid stamp (rib) on the boundary of a half-plane, i.e., $-1/2$.

But even under the restrictions adopted, the unique solution of problem (2.3) is obtained only after the general problem (1.3) is solved. Here we give only the final result:

$$\omega(z) = \frac{X_0(z)}{2\pi i} \int_{-1}^{+1} \frac{\omega_0(x)}{X_0^+(x)} \frac{dx}{x - z}, \quad X_0(z) = \sqrt{z^2 - 1}, \quad \omega_0(x) = -\pi\chi_1(x) - \lambda\chi_2(x). \tag{2.4}$$

Now, boundary-value problem (2.1) breaks up into two independent boundary-value problems for the new unknown subject to the conditions

$$\Gamma_1^+ + (-\chi_1 + \chi_3 + \chi_2 e^{-i\lambda})\Gamma_1^+ = F_1, \quad \Gamma_2^+ - (-\chi_1 + \chi_3 + \chi_2 e^{-i\lambda})\Gamma_2^+ = F_2, \tag{2.5}$$

and the shear field (1.2) is defined by the relation

$$\gamma_{13} - i\gamma_{23} = \gamma_{-1}^0 + e^{-i\omega}(\Gamma_1 + i\Gamma_2). \tag{2.6}$$

The solution of boundary-value problems (2.5) does not involve difficulties, and it is performed by the general rules for one-dimensional boundary-value Riemann problems [6]. We give the final formulas:

$$\begin{gathered}
\Gamma_k(z) = \frac{X_k(z)}{2\pi i} \int_{-1}^{+1} \frac{F_k(x)}{X_k^+(x)} \frac{dx}{x - z}, \quad k = 1, 2, \\
X_1(z) = (z-1)^{-1/2}(z-a)^{n-1/2}(z-b)^{-n}, \quad n = \lambda/2\pi, \\
X_2(z) = (z+1)^{-1/2}(z-a)^{n-1/2}(z-b)^{-n}, \\
F_1 = f_1 \cos\omega^+ - f_2 \sin\omega^+, \quad F_2 = f_1 \sin\omega^+ + f_2 \cos\omega^+.
\end{gathered} \tag{2.7}$$

Here f_1 and f_2 are the right sides of the first and second equalities of (1.3), respectively.

Thus, relations (2.4)–(2.7) are the results of the exact solution of the formulated problem (1.1) or (1.3) subject to condition (1.4).

3. Equivalent Problem. For rectilinear defects located on the real axis, the following principle of correspondence holds.

Let a solution of a certain boundary-value problem in the form (1.2) be known. Then the function

$$\tau_{13} - i\tau_{23} = \gamma_{-1}^0 + \gamma_2(z) + i\gamma_1(z) \tag{3.1}$$

is a solution of an equivalent boundary-value problem for which the boundary-value condition is obtained from the boundary-value condition of the initial problem by the substitution

$$\gamma_{13}^{\pm} \to \tau_{23}^{\mp}, \quad \gamma_{23}^{\pm} \to \tau_{13}^{\mp}, \quad \gamma_{13}^0 \leftrightarrow \gamma_{23}^0. \tag{3.2}$$

The validity of the correspondence is established from comparison of the limiting values of Cauchy type integrals that represent the functions $\gamma_1(z)$ and $\gamma_2(z)$.

Let us formulate a boundary-value problem that is equivalent to problem (1.1). In the new problem, the defect is a rigid, infinitely thin inclusion along the segment $|x| < 1$. The lower side is directly connected

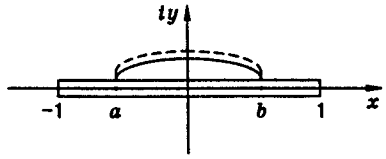

Fig. 2

to the main material as is the segment $\chi_1 = \chi_3 = 1$ on the upper side. The remaining part of the upper side ($\chi_2 = 1$) is connected to the main material via an elastic rib. In this case, the parameters c and μ in the formulas for the angle λ (1.1) determine the compliances of the rib and the main material, respectively. The contour of the defect is free from external forces, and rotations of the defect in its plane are prevented by internal moments (Fig. 2). A uniform external load is applied at infinity.

For this problem, the boundary condition is obtained from relations (1.1) by substitution (3.2), and the solution is

$$\tau_{13} - i\tau_{23} = \gamma_{-1}^{0} + e^{i\omega}(\Gamma_2(z) + i\Gamma_1(z)), \tag{3.3}$$

where the functions ω, Γ_1, and Γ_2 are given by formulas (2.4) and (2.7) in which the external strains are interchanged.

We note that the correspondence of (3.1) and (3.2) clarifies the reason for the impressive uniformity of the problems of cracks and rigid inclusions [7]. The field of the rigid inclusion (3.3) naturally inherits all properties of the field for cracks (2.6). In what follows, we shall not dwell on this except in special cases.

4. Asymptotic Expressions. The strain field (2.6) has three forms of asymptotic expressions: for the tips of the defect, the tips of the rib, and far away from the closely spaced tips of the defect and the rib, respectively. Let us consider them.

At the tips of the defect, the strains have a singularity typical of longitudinal-shear cracks [1, 2, 7]. For example, at the right tip, the asymptotic expression for the strain is as follows:

$$\gamma_{13} - i\gamma_{23} \cong -ik_3/\mu\sqrt{2\zeta_1}, \quad \zeta_1 \equiv z - 1, \quad |\zeta_1| \ll (1-b),$$

$$k_3 = \frac{\mu}{\sqrt{2\pi}}(1-a)^{n-1/2}(1-b)^{-n}\int_{-1}^{+1}\frac{F_1(t)}{X_1^{+}(t)}\frac{dt}{t-1}. \tag{4.1}$$

In particular, for $a = b = 0$, we obtain a common expression for a rectilinear cut: $k_3 = \mu\gamma_{23}^{0}$. In the other extreme case where a rib has a large length ($b \to 1$ and $a \to -1$), the stress-intensity coefficient increases infinitely:

$$k_3 = \mu(\gamma_{23}^{0} + \gamma_{\lambda 3}^{0})\frac{1-2n}{\sqrt{2}\cos\pi n}(1-a)^{n-1/2}(1-b)^{-n} + O(1+a) + O(1-b). \tag{4.2}$$

In the vicinity of the left tip of the defect, relations similar to equalities (4.1) and (4.2) hold. At the tips of the defect, the solution of the equivalent problem (3.3) has an asymptotic representation typical of a rigid inclusion:

$$\tau_{13} - i\tau_{23} \cong s_3/\mu\sqrt{2\zeta_1}. \tag{4.3}$$

Here s_3 is the stress-intensity coefficient, which coincides with values of k_3 after substitution (3.2).

In the vicinities of the rib tips, it is necessary to take into account the logarithmic singularity of the function $\omega(z)$. In this case, on the upper free side, the strains are continuous at both tips of the rib. In the vicinity of the right end of the rib from below, the strains are described by the asymptotic relations

$$\gamma_{13} - i\gamma_{23} \cong \frac{A-iB}{\zeta_2^{-2n}}, \quad \zeta_2 \equiv z - b, \quad |\zeta_2| \ll (1-b), \quad A - iB = \lim_{z\to b}(z-b)^n(\Gamma_1 + i\Gamma_2). \tag{4.4}$$

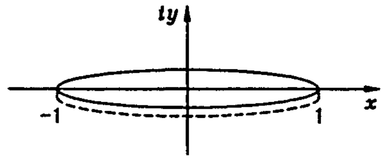

Fig. 3

In particular, for zero length of the rib ($a = b$), the intensity coefficient $A - iB$ in the asymptotic relation (4.4) vanishes, and in the opposite limit of a great length of the rib ($b \to 1$ and $a \to -1$), we obtain

$$A - iB = \frac{(\gamma^0_{\lambda 3} - \gamma^0_{23})(1-n)}{2\sin n\pi}(b+1)^{-1/2}(b-a)^{n-1/2}$$
$$- i\frac{(\gamma^0_{\lambda 3} + \gamma^0_{23})(1-2n)}{2\cos n\pi}(b-1)^{-1/2}(b-a)^{n-1/2} + O(1-b) + O(1+a). \tag{4.5}$$

The solution of the problem of a rigid inclusion (3.3) has continuous strains from below, and on the upper side near the right tip of the elastic rib, we have

$$\tau_{13} - i\tau_{23} \cong (B - iA)/\zeta_2^{-2n}. \tag{4.6}$$

The asymptotic relations for the strains at the left tips of the rib have the same structure but the order of the singularity is $(2n-1)$. In this case, in the limit of an absolutely rigid rib for solution (2.6) [and a crack for (3.3)] the asymptotic relations (4.4) and (4.6) describe a standard singularity of order $(-1/2)$, according to the restriction of Sec. 3. In the other extreme case of infinitesimal rigidity of the reinforcing rib ($n \to 0$), solution (2.6) becomes a solution for a crack with free sides, and the corresponding intensity coefficient in the asymptotic relation for the left tip of the rib vanishes. In the general case, the proof of this fact is rather cumbersome. Below, it is illustrated by an example.

Finally, asymptotic expressions of the third type describe the strain field at a distance from the closely spaced tips of the defect and the elastic rib. For example, at the right tips we obtain the asymptotic equalities

$$\gamma_{13} - i\gamma_{23} \cong e^{i\lambda/2}\left(c_{11}\zeta_3^{-1/2-n} + ic_{12}\zeta_3^{-n}\right),$$
$$c_{11} = \lim_{z\to 1}\{(z-1)^{n+1/2}\lim_{b\to 1}\Gamma_1(z)\}, \quad \zeta_3 \equiv z - 1, \tag{4.7}$$
$$c_{12} = \lim_{z\to 1}\{(z-1)^n\lim_{b\to 1}\Gamma_2(z)\}, \quad (1-b) \ll |\zeta_3| \ll 1,$$

and for the vicinity of the left tips, we have the relation

$$\gamma_{13} - i\gamma_{23} \cong e^{i\lambda/2}\left(c_{21}\zeta_4^{-1/2+n} + ic_{22}\zeta_4^{-1+n}\right), \quad \zeta_4 = z+1, \quad (1+a) \ll |\zeta_4| \ll 1. \tag{4.8}$$

The dependences of c_{21} and c_{22} on the functions $\Gamma_1(z)$ and $\Gamma_2(z)$, respectively, are similar to the dependences for c_{11} and c_{12}. The factor c_{22} vanishes in the limit $n \to 0$ as does the stress-intensity coefficient near the left tip of the rib.

5. Example. Let a reinforcing rib occupy the entire lower side of the defect, i.e., $b = -a = 1$. Appropriate boundary-value conditions are obtained from equalities (1.3) or (2.5) for $\chi_1 = \chi_3 = 0$, $\chi_3(x) \equiv 1$, and $\omega(x) = -\lambda/2$. Integrals (2.7) are calculated in elementary functions, and the strain field (2.6) near this defect (Fig. 3) has the form

$$\gamma_{13} - i\gamma_{23} = \gamma^0_{-1} + \frac{i(\gamma^0_{23} + \gamma^0_{\lambda 3})}{1 + e^{i\lambda}}\{1 - (z - 2n)X_1(z)\}$$
$$+ \frac{i(\gamma^0_{23} - \gamma^0_{\lambda 3})}{1 - e^{i\lambda}}\{1 - (z + 1 - 2n)X_2(z)\}, \tag{5.1}$$

where the functions $X_1(z)$ and $X_2(z)$ are defined by formulas (2.7) for $b = -a = 1$. The field (5.1) has only the third type of asymptotic expressions (4.7) and (4.8). In the limit of an infinitely rigid rib ($\lambda = \pi/2$ and $n = 1/4$), equality (5.1) is a solution of the primal mixed boundary-value problem. But the highest order of the singular terms occurs for weak reinforcement. For a small value of n, expression (5.1) has the following structure of the main singular terms:

$$\gamma_{13} - i\gamma_{23} = \gamma_{13}^0 - \frac{iz\gamma_{23}^0}{\sqrt{z^2-1}} - 2\gamma_{13}^0 n(z-1)^{-n}(z+1)^{n-1} + O(n). \tag{5.2}$$

Here the first two terms on the right side give an exact solution for the crack with free sides, and the last two terms determine the strain redistribution under the influence of a rib of small rigidity. From equality (5.2) it is evident that the stress-intensity coefficient for the left tip of the defect is proportional to the small parameter n and, hence, in the limit $n \to 0$, it vanishes. The stress redistribution postulated by solution (5.2) is due to the fact that, by the data given, the rotations of the rib in its plane are prevented by the internal moment. The possible rotations of the rib are similar to the additional field γ_{13}^0. It remains to calculate the internal moment that holds the rib. For the complete solution (5.1), the required quantity m has the form

$$m = -\mathrm{Im} \oint z(\gamma_{13} - i\gamma_{23})\, dz = \frac{\pi\gamma_{23}^0}{4} \sin\lambda(8n^2 - 1) - \frac{\pi\gamma_{13}^0}{2}\left(4n^2\cos\lambda + \sin^2\frac{\lambda}{2}\right),$$

where the path of integration encloses the defect, and, hence, the parameter m is equal to the principal moment acting on the defect.

6. Paradox of the Primal Mixed Boundary-Value Problem. Solution (5.1) is incompatible with the variational symmetry of the antiplane deformation equations about the group of spatial shears. To derive this contradiction, we consider the variational symmetries of the problem. Using the general methods described, e.g., in [8, 9], we obtain

Statement. *The vector field $\xi_k\partial_k$ generates zero divergences of the form*

$$\partial_j\left(\xi_j L - \frac{\partial L}{\partial w_j}\xi_k w_k\right) = 0, \quad w_j = \partial_j w \tag{6.1}$$

for the Lagrangian of the static antiplane strain of a homogeneous isotropic media. Here w is the scalar of displacements if the generatrices satisfy the Cauchy–Riemann relations

$$\partial_1\xi_1 - \partial_2\xi_2 = 0, \quad \partial_1\xi_2 + \partial_2\xi_1 = 0. \tag{6.2}$$

This statement is proved by substitution of equalities (6.1) and (6.2) into the condition of variational symmetry [8, p. 328] and can be extended to orthotropic media with appropriate modification of the second equation of (6.2).

Divergences (6.1) are conveniently written in integral form with allowance for complex representations for the fields. Omitting calculations, we write the final result as

$$I[\Phi] = \frac{i\mu}{2}\oint_S \Phi(z)(\gamma_{13} - i\gamma_{23})^2\, dz, \tag{6.3}$$

where $2\Phi(z) \equiv \xi_1 + i\xi_2$ is the complex generatrix of the vector field generating variational symmetry and S is an arbitrary closed contour. If the contour S is in the region of holomorphism of the field $(\gamma_{13} - i\gamma_{23})^2$, then $I[\Phi] = 0$ is a zero divergence generated by the field with a generatrix $\Phi(z)$. In particular, spatial shears, as usual, lead to invariant J integrals with components J_1 and J_2 and a generatrix $\Phi(z) = 1$, related by the equality

$$I[1] = J_1 - iJ_2. \tag{6.4}$$

The rotations and dilatations are responsible for the conservation and invariance with respect to the shape of the contour S of moment integrals of the form

$$I[z] = N + iM = \frac{i\mu}{2}\oint_S z(\gamma_{13} - i\gamma_{23})^2\, dz. \tag{6.5}$$

As is known, formula (6.4) was also obtained as an invariant integral of the first kind by another method [10]. The existence of invariants related to the groups of rotations and dilatations in the three-dimensional and planar theory of elasticity is established in [11]. Formula (6.5) specifies their form for the antiplane problem. The general expressions (6.1)–(6.3) describe the quantities conserved for an arbitrary conformal mapping. They are not reduced to invariant integrals of the second kind [10] and allow one to determine the structure of the strain field at any point of a body containing a linear defect (by definition, the contours of the defect are free from external forces and, hence, logarithmic singularities do not arise). Omitting obvious reasoning, we write the final result for the case where the contour S encloses one singular point of the field $(\gamma_{13} - i\gamma_{23})^2$ with an affix $z = z_0$:

$$(\gamma_{13} - i\gamma_{23})^2 = f(z) - \frac{1}{\mu\pi} \sum_{k=1}^{\infty} I_k (z - z_0)^{-k}, \quad I_{k+1} \equiv I[(z - z_0)^k]. \tag{6.6}$$

Here $f(z)$ is the regular portion of the strain square in the vicinity of z_0.

What conclusions regarding the strain field can be drawn from expansion (6.6)? Generally, extraction of the square root leads to a rather complex and useless expression. We consider the most important particular case that follows from the following two restrictions:

1) The equalities are considered asymptotically;

2) We restrict ourselves to linear defects about which the displacements are finite and the energy is integrable.

In this case, the main part of expression (6.6) contains only one term, which uniquely determines the asymptotic strain relations

$$\gamma_{13} - i\gamma_{23} \cong \pm i \sqrt{\frac{J_1 - iJ_2}{\pi\mu(z - z_0)}}. \tag{6.7}$$

Thus, equality (6.7) establishes the general form of the asymptotic strain distribution at the tip of a linear defect that corresponds to the variational symmetry of the equations about spatial shears. In particular, asymptotic relations for a crack (4.1) and rigid inclusions (4.3) coincide with the general form. Asymptotic relations with other singularities, for example (4.7) and (4.8), are incompatible with the variational symmetries of (6.1)–(6.3). We come to a contradiction.

7. Solution of the Paradox. The general case of solutions (2.6) and (3.3), where $b \neq 1$ and $a \neq -1$, meets the requirements of variational symmetry because the asymptotic relations for the tips of the defects coincide with expression (6.7). The paradoxical properties of the field (5.1) are due to the fact that it corresponds to the special state of instantaneous rearrangement of solution (2.6) with the asymptotic expression for cracks (4.1) to solution (3.3) with the asymptotic expression for rigid inclusions (4.3) in the hypothetical process of continuous variation in the length of the rib at the moment when $b = -a = 1$. Avoiding a formal analysis of this singularity, we determine whether the state (5.1) can be realized under conditions of limited strength of the material. The answer to this question is negative.

We assume that a certain hypothetical process, as mentioned above, makes possible continuous variation in the length of the reinforcing rib within the limits of applicability of solution (2.6). For definiteness, we consider an absolutely rigid rib in the vicinity of the right tip of a defect (Fig. 4a). As the parameter b (Fig. 4b) increases, the values of k_3 also increases infinitely by formula (4.2). When it reaches a certain critical level k_{3c}, one of the two following variants is possible: slow steady crack growth begins if the strength of the main material is sufficiently small compared to the adhesion strength (Fig. 4c) or the rib separates from the main material if the inverse ratio of these parameters is inverse (Fig. 4d). A similar picture emerges for solution (3.3). The slow growth of the crack (in a certain hypothetical process) on the upper side (Fig. 5a and b) ceases when the parameter b in the asymptotic relation (4.6) reaches the adhesion strength level. Rapid supercritical growth of this crack leads to one of the two new steady states (Fig. 5c and d), which are similar to those shown in Fig. 4c and d.

Thus, under conditions of limited strength, the state (5.1) cannot be attained from solution (2.6) or from solution (3.3) because it corresponds to the subcritical phase of a brittle crack. The defect configurations

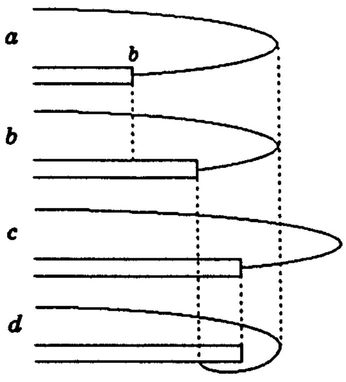

Fig. 4

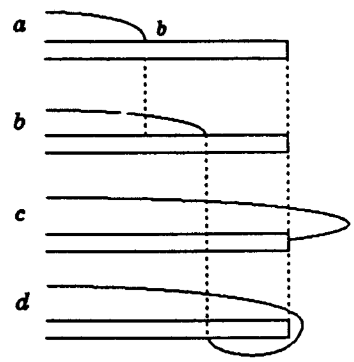

Fig. 5

shown given in Fig. 5a and b are unstable. As $b \to 1$, they become the other forms shown in Fig. 4c and d and Fig. 5c and d. Solutions of type (5.1) can be used only to estimate the strain field far away from closely spaced tips of the defect and the rib. However, they can be employed to estimate the intensity coefficient from comparison with the asymptotic relations for the near zone. For example, for an absolutely rigid rib and the right tips we have the following asymptotic relations:

$$\gamma_{13} - i\gamma_{23} = \begin{cases} \dfrac{1-i}{4\sqrt[4]{2}}(\gamma_{13}^0 + \gamma_{23}^0)\zeta^{-3/4} + \dfrac{3(1+i)}{4\sqrt[4]{8}}(\gamma_{13}^0 - \gamma_{23}^0)\zeta^{-1/4} & [(1-b) \ll \zeta \ll 1], \\ -\dfrac{i(1-b)^{-1/4}}{2\sqrt[4]{2}}(\gamma_{13}^0 + \gamma_{23}^0)\zeta^{-1/2} & [\zeta \ll (1-b)]. \end{cases}$$

The stress-intensity coefficient can be obtained by coupling these asymptotic relations by one or another known method.

8. Conclusions.

1. We considered a number of mixed problems of the theory of longitudinal-shear cracks that originate from the problem of a plane with a rectilinear cut whose lower side is partly reinforced by an elastic rib (1.1).

2. The boundary-value problem (1.3) is solved using a special method consisting in diagonalization of the coefficient matrix by the orthogonal analytical matrix (2.2), which, in turn, is determined from an auxiliary boundary-value problem with condition (2.3).

3. An equivalent problem of a rigid inclusion is formulated, and its exact solution (3.3) is constructed.

4. The exact solution of problem (2.6) has three forms of asymptotic strain relations (4.1), (4.4), and (4.7). The main properties of these relations are discussed.

5. A primal mixed problem whose solution (5.1) has only one form of asymptotic expressions (4.7) and (4.8) is considered in detail.

6. The general form of the asymptotic strain relations due to the group of spatial shears (6.7) is constructed from an analysis of the variational symmetries of the Lagrangian of antiplane deformations (6.1). A paradox of the primal mixed problem is formulated. A solution for this paradox is proposed based on the fact that paradoxical solutions cannot be realized under conditions of limited accuracy.

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